What the Past Tells about the Future:
Historical Prices in the Durable Goods Market

Zheng Gong†, Jin Huang‡ and Yuxin Chen‡

†University of Toronto
‡NYU Shanghai

Abstract

We investigate the dynamic pricing strategy of a durable good monopolist in a new setting that assumes away perfect consumer information on arrival time and historical prices. We first show that when consumers with heterogeneous tastes are all uninformed of historical prices, the monopolist charges a high price for most of the time and periodically holds low-price sales. Then we consider the case in which a small fraction of consumers become informed about historical prices (such as price tracker users). We find that in equilibrium the seller lowers the high price and keeps it for a shorter period in each price cycle. With the presence of price trackers, the monopolist profits less while consumer welfare increases. We contribute to the literature by illustrating the informational role of historical prices and understanding how price tracking websites affect the market equilibrium. Our findings also provide microfoundations of price stickiness and have important managerial implications for sellers and platforms on historical price disclosure.

Keywords: durable goods; dynamic pricing; historical prices; price trackers
1 Introduction

When we talk about price comparison prior to a purchase, normally it refers to comparing current prices of alternatives. Sellers rarely disclose the price history of their products, as a result, it is hard for consumers to conduct a temporal price comparison for one product. From the supply side, we are interested to know if disclosing the price history affects the seller’s pricing strategy and profit. From the demand side, we want to answer if consumers can use past price trajectory to predict future prices and make a better decision in the timing of purchase.

To the best of our knowledge, this paper is the first to study historical prices and address those questions. They are especially relevant nowadays as it is easier to acquire past price data in the era of digitalization and big data. For example, price trackers, websites that provide products’ historical prices, have come on the market. This paper also aims at explaining how the presence of price trackers changes the market equilibrium.

When a price tracker user decides whether to complete a purchase today or hold off, how can he make use of the historical information? A myopic prediction might be that the future pricing pattern simply follows the historical pattern. Our main intuition agrees on this prediction, however, with a rationale behind.

We model a durable good monopolist setting prices dynamically over an infinite time horizon. A new cohort of consumers enters the market at every instant and are interested in making a once for all purchase either immediately or after a delay. Consumers are heterogeneous in their tastes for the product and as such their willingness to buy are different.

Intuitively, the monopolist will conduct an intertemporal price discrimination by changing price from high to low in each price cycle repeatedly over time. When all the consumers are uninformed of the product’s price history, consumers do not know where they stand in

---

1 Among all price trackers, CamelCamelCamel (hereinafter referred to as CCC) has become one of the most popular for the better part of a decade. It is an Amazon price tracker that provides price history charts for products sold by Amazon. To use it, consumers simply need to copy and paste the URL of an Amazon item at the top of CCC’s site, and then it will show the item’s “Price History” chart.
a price cycle. The monopolist will set one single high price as a pooling equilibrium for a
certain length of time. This high price will be such that all high-valuation consumers who
arrive during this time period are just indifferent from purchasing immediately or delaying
the purchase to future sales, and low-valuation consumers wait in the market for the sales.
When the high-price period ends, enough low-valuation consumers have been accumulated.
The monopolist will hold a sale at a lowest price to clear the market. The high-price period
and the sale at the end constitute a price cycle. Right after each sale, a new price cycle
restarts.

The mechanism behind price cycle formation is the key to understand the informational
role of historical prices. When a consumer (for example, a price-tracker user) becomes in-
formed of historical prices, only the price data points in the current price cycle are necessary
in predicting the future price. This is because from where he is in the current cycle, the con-
sumer could infer the number of low-valuation predecessors who are waiting in the market.
Hence, they are able to perfectly predict when the next sale will take place, as the timing of
holding a sale is determined by the stock of waiting consumers. Compared to consumers
who do not know historical prices, price tracker users may strategically wait for the sale
instead of purchase immediately. Because of their strategic waiting behavior, we find that
the monopolist lowers the high price and shortens the high-price period, interpreted as the
informational rent taken by the informed consumers. Since average price is lower and sales
are more frequent with more price tracker users, the monopolist’s profit decreases, while the
welfare of consumers increases. This implies that it is rational for sellers not to disclose their
price history.

Note that much of what we have discussed so far is mostly applicable to the durable
goods market (or, non-frequently-purchased goods market). As opposed to non-durable
goods like fruit or clothing, the product value is stable across time. Consumers are normally
interested in making a once-for-all purchase and flexible at purchase timing. Hence, a mo-
nopolist is competing with future versions of himself, as forward-looking consumers may
choose to wait if they believe prices will be lower in the future.

This paper contributes to the literature by illustrating the informational role of historical prices and price trackers for the first time. Our paper is related to several streams of literature in economics and management. Conlisk et al. (1984) is the most relevant to our work. They consider a dynamic pricing model of a monopolist who faces strategic consumers with heterogeneous valuations to the good, and they show that the optimal pricing path of the monopolist is cyclical. Sobel (1984) generalizes this framework to more than one seller, and shows that under oligopolistic competition, the cyclical pattern of sales still exists. Sobel (1991) revisits this problem and characterizes all feasible payoffs that the monopolist can attain with or without commitment to the pricing scheme when agents are patient enough. Villas-Boas (2004) focuses on a monopolist’s behavioral price discrimination, discriminating according to a customers purchasing history, which generates a cyclical pricing pattern. This paper is also related to the revenue management literature which discusses a firm’s dynamic pricing problem with strategic forward-looking consumers (Gallego and Van Ryzin, 1994, 1997; Su, 2007; Levin et al., 2009).

All these previous papers assume that consumers have perfect information. To capture a more realistic environment, our model assumes away perfect consumer information. Instead, we consider a benchmark setting where consumers do not know their entry time in the market and cannot observe historical prices. This information friction leads to a different shape of equilibrium pricing scheme compared to the one obtained by Conlisk et al. (1984). Moreover, the new pricing pattern exhibits price stickiness which has been well documented in the empirical literature, and as such, our theory provides microfoundations of price stickiness and menu cost.

The remainder of this paper is organized as follows. In Section 2, we introduce the model setting. Section 3 analyzes the benchmark case where all consumers are uninformed about the price history and their entry time. In Section 4 we deviate from the benchmark setting by assuming that a fraction of consumers become informed about historical prices, and we
solve for the new market equilibrium and compared it with the benchmark market outcome. Section 5 provides the welfare implications of the presence of price tracking websites. In Section 6, we add an extension to the model to consider the market outcome when the monopolist has full commitment power. Finally we conclude in Section 7.

2 Overview of the Model

We consider a durable good market with a monopolist (seller, she). The market operates over a continuous-time infinite horizon with a discount factor $r \in [0,1]$ that is common to all parties. The monopolist produces the product at a constant marginal cost of production, which is normalized to be zero without loss of generality. The monopolist sets the product price dynamically, and she has no means of credibly committing to a predetermined pricing scheme. Moreover, we assume that the monopolist has to maintain any price level for at least $L$ period length due to menu cost or any kind of regulatory cost.

A flow of atomic consumers (buyers/buyer, they/he) continuously enter the market at every instant $t$, where $t \in [0,\infty)$. Each consumer is interested in buying one unit of the product and the product cannot be resold or rented out. Consumers are heterogeneous in two dimensions. First, a share $\alpha \in [0,1)$ of consumers are of high-valuation and a share $1 - \alpha$ are low-valuation. High-valuation consumers value the product at $v_h$ and low-valuation consumers value it at $v_l$, where $v_h > v_l > 0$. Secondly, for any valuation type, with a probability of $\gamma (0 \leq \gamma \leq 1)$ a consumer is informed about the product’s price history which is defined as a complete record of all the past prices at any instant.

Upon his entry, every consumer faces the current price set by the monopolist, and he decides whether to purchase or not. If he buys the product, afterwards he leaves the market forever. If he decides not to buy, he can stay in the market and buy the product at any time in the future. A consumer who never makes a purchase receives a zero utility. We assume that when a consumer is indifferent between buying now and buying later, he buys now; when
a consumer is indifferent between buying and not buying, he buys. Following the past literature, we look for a pure strategy equilibrium and when there are multiple equilibria, we characterize the monopolist’s most preferred equilibrium.

In the literature on durable goods market, perfect consumer information is typically assumed. In our model, we consider a more realistic set-up which assumes away perfect consumer information. In Section 3, we first present the solution to the monopolist’s problem when all consumers are uninformed about the history, thus ignorant of the relative timing at which they enter the market. In Section 4, we relax the information asymmetry, and analyze the market equilibrium when some consumers (for example, the price-tracker users) become informed of historical prices of the product.

3 Benchmark: No Informed Consumers

Consumers of time $t$ are not informed about past prices and purchase records, instead they only know the price at $t$. In this section, we analyze the market equilibrium when there are no informed consumers or before the presence of price trackers. We start by characterizing the properties that the equilibrium pricing pattern involves, and then we will analytically solve for the equilibrium price and the price-cycle length.

3.1 Properties of the Equilibrium Price Cycles

Following similar mechanisms illustrated in Conlisk et al. (1984), the equilibrium in our model also involves a cyclic pricing pattern. The intuition is that, in order to prevent current consumers from waiting for the sales, the monopolist normally charges a price that is just low enough to sell immediately to the fraction of high-valuation consumers, but too high for low-valuation consumers. Over time, more and more low-valuation consumers are accumulated in the market. When the accumulated pool is large enough, the monopolist will drop
the price to their reservation level to clear the market\textsuperscript{2}. After all stocked past consumers are cleared out of the market, the monopolist repeats the pattern starting with a high price.

In the models with perfect consumer information, in equilibrium, at every period/instant, new consumers could predict how long they need to wait for the sale. The closer their entry time is to the sale, the lower the price has to be to prevent waiting. In our model, consumers only observe the current price when they enter the market, without knowing the exact time of entry. When the high-valuation consumers observe a price higher than the low reservation value \( v_l \), they hold an even prior about where they locate in the whole interval of non-sale period. Hence, even when the sale is coming soon, high-valuation consumers have no knowledge about it and may just buy right away at a fairly high price. As a result, the equilibrium differs from the classic literature in both the pricing pattern within a cycle and the cycle length. The following lemmas summarize some key properties of the equilibrium pricing pattern:

**Lemma 1** Let \( P \) denote the set of all different prices that the monopolist charges over time in equilibrium. The minimum value of \( P \) is \( v_l \), and low-valuation consumers only buy the product at this price.

The proof goes as follows: if \( \min P > v_l \), as it is greater than the reservation price of low-valuation consumers, low-valuation consumers will not buy but stay in the market. Let \( \bar{p} \) denote the maximum value of \( P \). There exists a \( \tilde{t} \) such that for any \( t \geq \tilde{t} \), the monopolist could deviate from any current price to \( v_l \) for length \( L \). By doing that, she deviates to clearing the market at \( v_l \) price and gains a higher profit. \( \tilde{t} \) is pinned down by:

\[
\alpha \bar{p} \int_0^L e^{-rt} dt = (1 - \alpha)v_l \tilde{t} + v_l \int_0^L e^{-rt} dt \tag{1}
\]

If \( \min P < v_l \), a profitable deviation for the monopolist is to increase all prices that are

\textsuperscript{2} Following Conlisk et al. (1984), we define that whenever the monopolist sets the price low enough to sell to low consumers, she "holds a sale".
lower than \( v_l \) to \( v_l \). It will not affect consumers’ actions because for consumers who purchase at a price higher than \( v_l \), they will still buy immediately and not wait instead because all the future prices are the same or higher after the deviation; for consumers who buy previously at a price that is lower than \( v_l \), now they will still buy after the monopolist raises the price, as \( v_l \) is the current lowest price. Hence, the deviation is profitable, and \( \min P < v_l \) cannot constitute an equilibrium.

We further specify another important property of the monopolist’s optimal pricing strategy in equilibrium in Lemma 2:

**Lemma 2** The set \( P \) contains only two elements, that is, \( P = \{p^*, v_l\} \) and \( p^* \geq v_l \). In equilibrium, high-valuation consumers never delay their purchase and always buy immediately following their entry at either price \( p^* \) or price \( v_l \).

From Lemma 1, we know that the lowest price is \( v_l \). Any price higher than \( v_l \) will not be accepted by low-valuation consumers. Let \( H \) denote the set of prices high-valuation consumers are charged and will buy at in the equilibrium. As consumers prefer to pay more to avoid costly waiting, \( H \) contains at least one more price that is higher than \( v_l \). If \( H \) contains more than two different prices, let \( H = \{p_1, p_2, \ldots, p_N, v_l\} \), and w.l.o.g \( p_1 > p_2 > \cdots > p_N > v_l \). Then for the seller, a profitable deviation is to replace all \( p \in P \) and \( p \neq v_l \) to \( p_1 \). The high-valuation consumers will buy at \( p_1 \) since they cannot tell where they currently are in a price cycle, and low-valuation consumers will only accept price \( v_l \). Hence, this is a deviation that yields a higher payoff.

With Lemma 1 and Lemma 2, we can fully characterize the shape of the optimal pricing scheme as follows:

**Proposition 1 (Shape of the optimal pricing scheme)**

The optimal pricing scheme is cyclical and fully captured by \( (p^*, T^*) \), where \( T^* \geq L \) is the length of \( p^* \). We denote \( p^* \) as the **normal price**, and \( v_l \) as the **sale price**. Each cycle starts with the normal price \( p^* \) for \( T^* \) length of time, and ends at the sale price \( v_l \) lasting for \( L \) length of time. The
equilibrium price cycle length is $T^* + L$. \(^3\)

To elaborate: early in each price cycle, the high-valuation consumers buy immediately after they enter the market at the normal price, while low-valuation consumers wait in the market. As there are more and more low-valuation consumers accumulated in the market over time, the incentive for the monopolist to hold a sale at the sale price becomes higher and higher since it is better to clear out the market and gain the revenue from the accumulated group as early as possible. Therefore, the monopolist will start a sale after $T^*$ time of the normal price. The accumulated low-valuation consumers from previous $T^*$ time will buy immediately at the beginning of the sale. After clearing out all accumulated consumers, the sale still lasts for a time length $L$, which is the shortest time a price remains unchanged for. After that, a new price cycle starts in the same way.

3.2 Solving for the Equilibrium with No Informed Consumers

The monopolist’s problem is to dynamically maximize the present value of total future profits by setting the optimal $p^*$ for an optimal length $T^*$. Let $\Pi(p, T)$ denote the present value of the seller’s profit stream, where $p$ and $T$ correspondingly denote the candidate values for the normal price and the period length of the normal price in equilibrium:

$$
\Pi(p, T) = \left[ \alpha p \int_0^T e^{-rt} dt + (1 - \alpha)Tv_l e^{-rT} + v_l \int_T^{T+L} e^{-rt} dt \right] \left( \sum_{i=0}^{\infty} e^{-ir(T+L)} \right)
$$

$$
= \frac{\alpha p(1 - e^{-rT}) + (1 - \alpha) v_l r T e^{-rT} + v_l (e^{-rT} - e^{-r(T+L)})}{r(1 - e^{-r(T+L)})}
$$

(2)

Within each price cycle, the profit comes from three parts: during time $[0, T)$, only high-valuation consumers buy at the normal price $p$. At $T$, the price changes to $v_l$, and all low-valuation consumers who enter between $[0, T)$ will immediately buy at time $T$. Finally the

\(^3\) The belief off the equilibrium path is defined in such a way that consumers will not purchase if they observe a price different from the equilibrium normal price $p^*$ or $v_l$, but they will always purchase if the current price is $v_l$. 

9
sale price $v_l$ lasts for $L$ length and all consumers entering during this period of time will directly buy at the sale price. The three parts constitute the monopolist’s profit from one price cycle, and the monopolist maximizes the sum of all the discounted future profits from repeated price cycles.

The monopolist chooses the optimal $(p, T)$ given two constraints. First, when the high-valuation consumers enter the market and see the normal price, they prefer to buy immediately rather than waiting for the sale. This requires:

$$v_h - p \geq \frac{1}{T} \int_0^T (v_h - v_l) e^{-rt} dt = (v_h - v_l) \frac{1 - e^{-rT}}{rT}$$

(3)

Secondly, the monopolist has no incentive to advance or delay the sale at any moment in equilibrium. Lemma 3 presents a sufficient condition for this IC condition on the monopolist to hold:

**Lemma 3** For any given $p$, $\Pi(p, T)$ is globally continuous, differentiable and concave on $T - p$ diagram. And a sufficient condition for the IC constraint on the monopolist to be hold is $\partial \Pi(p, T) / \partial T = 0$.

For any given $p$, let $T^*$ be the normal-price period length that solves equation $\partial \Pi(p, T) / \partial T = 0$. Let $t$ denote the point of time in a price cycle before when the monopoly only charges the normal price. We prove that for any $t < T^*$, at $t$, the monopolist prefers to wait for at least $dt$, which goes to zero in the limit, to start a sale rather than doing it right away; for any $t > T^*$, the monopolist prefers to advance the sale by at least $dt$ than doing it at $t$. Let $\pi_t$ denote the profit function of the monopolist holding a sale at $t$:

$$\pi_t = (1 - \alpha)tv_l + \int_0^L v_l e^{-rs} ds + e^{-rL}\Pi(p, T^*)$$

(4)
And the profit from holding a sale at $t + dt$ is:

$$
\pi_{t+dt} = \int_0^{dt} \alpha pe^{-rs} ds + (1 - \alpha)(t + dt)v_l e^{-rdt} + \int_{dt}^{dt+L} v_l e^{-rs} ds + e^{-r(L+dt)} \Pi(p, T^*)
$$

(5)

Let $\Delta_t = \pi_{t+dt} - \pi_t$. Note that in the expression of $\Delta_t$, the only term that contains $t$ is $(1 - \alpha)t v_l (e^{-rdt} - 1) < 0$, therefore $\frac{\partial \Delta_t}{\partial t} < 0$ always holds. Moreover, recall that at $T^*$, $\frac{\partial \Pi(p, T)}{\partial T}|_{T = T^*} = 0$. By definition, $\Delta_t|_{t = T^*} = 0$ when we take limits as $dt$ goes to zero. Thus for any $t < T^*$, we have $\Delta_t > 0$ and at any $t > T^*$, we have $\Delta_t < 0$. Therefore, the monopolist has no incentive to deviate from the equilibrium $T^*$ by advancing or delaying the sale.

To bring all the pieces together, we can formalize the problem as a constrained optimization problem of the monopolist over $(p, T)$ space as follows:

$$
\max_{\{p, T\}} \Pi(p, T) = \frac{\alpha p (1 - e^{-rT}) + (1 - \alpha) v_l r T e^{-rT} + v_l (e^{-rT} - e^{-r(T+L)})}{r(1 - e^{-r(T+L)})}
$$

(6)

subject to:

$$
v_h - p \geq (v_h - v_l) \frac{1 - e^{-rT}}{rT}
$$

(7)

$$
(\alpha p - v_l) (1 - e^{-rL}) + (1 - \alpha) v_l [1 - rT - e^{-r(T+L)}] = 0
$$

(8)

On the $T - p$ diagram, (7) with equality can be rewritten as

$$
p = v_h - (v_h - v_l) \frac{1 - e^{-rT}}{rT}
$$

(9)

Function (9) is increasing and concave in $T$. The left-hand side of equation (8) is increasing and convex in $T^4$. As $T \to \infty$, (9) converges to $p = v_h$ and the price that satisfies (8) goes to infinity. Hence, as long as when $T = L$ (as we assume $L$ is the smallest length for any price), the price on (8) is lower than the value of (9), we have a non-empty feasible set

\footnote{A formal proof is in the Appendix A.}
of \((P, T)\) satisfying two constrains. The sufficient condition is:

\[
\frac{v_l}{\alpha} - \frac{(1 - \alpha)v_l[1 - rL - e^{-2rL}]}{\alpha(1 - e^{-rL})} \leq v_h - (v_h - v_l) \frac{1 - e^{-rL}}{rL}
\] (10)

Note that \(\frac{1 - e^{-rL}}{rL} < 1\) for all \(L > 0\). When \(v_h\) is large enough, the sufficient condition holds. As (8) implicitly defines a function \(p^{ICM}(T)\), we plug it into (6). Based on the chain rule, we have:

\[
\frac{d\Pi(p^{ICM}(T), T)}{dT} = \frac{\partial \Pi(p, T)}{\partial T} + \frac{\partial \Pi(p, T)}{\partial p} \frac{\partial p^{ICM}(T)}{\partial T} = (1 - \alpha)v_l \frac{1 - e^{-rT}}{1 - e^{-rL}} > 0
\] (11)

Therefore, the monopolist will increase both elements in \((p, T)\) along constraint (8) until constraint (7) is binding. Lastly, we also need to compare the profit in this equilibrium to the profit of a constant price setting where \(p = v_l\). Note that this is easy to compare as when \(T \to 0\), both (7) and (8) converges to \(p = v_l\). By (11), a constant \(p = v_l\) pricing scheme must be giving us a lower payoff. To sum up, Proposition 2 describes the equilibrium when there are no informed consumers in the market:

**Proposition 2 (Equilibrium with no informed consumers)**

When there are no informed consumers, in the region \(T \geq L\), the equilibrium pricing scheme \((p^*, T^*)\) is uniquely pinned down by the crossing point of the incentive compatibility constraint on the monopolist and the binding incentive compatibility constraint on high-valuation consumers.
Figure I: Equilibrium pricing scheme with no informed consumer

We provide a numerical example of the equilibrium in Figure I. We take parameters \((v_h = 10, v_l = 4, \alpha = 0.8, r = 0.02, L = 2)\). The blue curve is the upper boundary of IC constraint on high-valuation consumers, and the red curve is the IC constraint on the monopolist. The crossing point of red and blue curve determines the equilibrium \((p^*, T^*)\).

4 The Model with Informed Consumers

In this section, we assume that within both high-valuation and low-valuation consumers, a fraction \(\gamma (\gamma > 0)\) of them are informed, who are possibly price-tracker users and could observe the product’s historical prices. More specifically, now there are four types of consumers: \{High, Low\} \times \{Informed, Uninformed\}, denoted by \{HI, HU, LI, LU\}.

Following the same logic applied to Lemma 1, the lowest price in the equilibrium cannot be lower than \(v_l\). Hence, both LI and LU types of consumers will wait until the sale
starts. HU consumers have no information about the timing of their arriving and hold an even prior belief on the timing of the next sale. They will buy immediately when they enter the market if the price is low enough to make the purchase yield a higher payoff than the expected payoff of waiting to buy at the sale. HI consumers are able to predict the timing of the next sale as they could infer the number of accumulated low-valuation consumers from historical prices. Hence, HI consumers might wait if the next sale is close to their entry time. Intuitively, it could possibly be optimal for the monopolist to set prices stepwise, especially in the time region close to the sale, to prevent informed high-valuation consumers from waiting. This step-function equilibrium shape is more likely to occur if the fraction of informed consumers is large. Otherwise, if $\gamma$ is small enough, in equilibrium the monopolist only charges one normal price. Let $(\hat{p}, \hat{T})$ denote the equilibrium pricing scheme with informed consumers. The following proposition formalizes it:

**Proposition 3** With a small enough $\gamma$, any pricing scheme with more than two prices besides $v_l$ cannot constitute an equilibrium. The equilibrium set of prices can only be either \{\hat{p}, v_l\} where $\hat{p} > v_l$, or \{v_l\}.\(^5\)

Here is the intuition: if the seller wants to lower price to capture the informed high-valuation consumers, in the meantime, the revenue from those uninformed high-valuation consumers becomes lower. When $\gamma$ is small, the gain cannot offset the loss. As such, the seller prefers not to lower price to capture any informed consumers at the moment of their entry. We also provide the complete proof in Appendix A.

To better compare results of the two models, we only focus on the market equilibrium with a small enough $\gamma$. The monopolist maximizes the present value of all future profits, under two constraints: HU consumers prefer to buy immediately when they arrive the market; and the monopolist has no incentive to advance or delay the sale. For any pricing scheme

\(^5\) The assumption on $\gamma$ being small is consistent with the reality. By the end of year 2016, the users of the biggest Amazon price tracker CamelCameCamel reached 1 million, while Amazon owned more than 300 million users by then.
there exists a threshold $N(p)$ such that, for HI consumers who enter the market during $[T - N(p), T]$ of a price cycle, they prefer to wait for the sale rather than to buy at the current price. $N(p, T)$ is defined as follows:

$$N(p) = \frac{1}{r} \ln \frac{v_h - v_l}{v_h - p}$$

For simplicity of notation, hereafter we let $N$ denote the period length during which informed high-valuation consumers enter the market and they wait for the sale. The objective function for the monopolist is as follows:

$$\hat{\Pi}(p, T) = \alpha (1 - \gamma) \frac{p(1 - e^{-rT}) + \alpha \gamma p(1 - e^{-r(T-N)}) + [T(1 - \alpha) + N\alpha \gamma]v_l e^{-rT} + v_l(e^{-rT} - e^{-r(T+L)})}{r(1 - e^{-r(T+L)})}$$

The numerator presents the profits from one price cycle. The first two terms are profits from HU consumers who enter the market during the normal-price period $T$, and from HI consumers who enter during the early $T - N$ period within a price cycle. The third term represents the profits from clearing out, at $t = T$, the accumulated low-valuation consumers LI and LU across period $T$ and HI consumers who wait in the market from $T - N$ to $T$. The last term represents the profits from consumers entering during the sale period which lasts for $L$.

The first constraint remains unchanged as in the benchmark. HU consumers will buy at the moment of entry rather than wait if:

$$v_h - p \geq (v_h - v_l) \frac{1 - e^{-rT}}{rT}$$

The second constraint on the monopolist’s incentive is different from the benchmark. In this case, we can no longer use the first-order condition of $\hat{\Pi}(p, T)$ with respect to $T$ to find the optimal choice on $T$. The reason is that, when the monopolist decides if she should

\[\text{Note that this constraint also guarantees that } N(p) < T. \text{ The reason is as follows: in the interval of } [T - N(p), T], \text{ the HI consumers will wait until } t = T \text{ and buy at price } v_l; \text{ if } N(p) \geq T, \text{ then for the whole interval of } [0, T], \text{ HI consumers will wait. But then a HU consumer will also wait no matter when he enters the market, which conflicts with (14). Hence, } N(p) < T \text{ has to hold.}\]
advance the sale or not, the cost she bears only comes from HU consumers paying at a lower price, whereas in the benchmark model the monopolist bears a cost from all high-valuation consumers paying less. The first-order condition is valid only for the latter case.

Given any \((p, T)\), we define \(\hat{W}(p, T)\) as the discounted future profits at the beginning of a sale:

\[
\hat{W}(p, T) = \left[ T(1 - \alpha) + N\alpha\gamma v_1r + v_1(1 - e^{-rL}) + \alpha(1 - \gamma)p(e^{-rL} - e^{-r(T+L)}) + \alpha\gamma p(e^{-rL} - e^{-r(T-N+L)}) \right] \frac{1}{r[1 - e^{-r(T+L)}]}
\]

(15)

To guarantee that the seller has no incentive to advance or delay the sale, we should have:

\[
\frac{\alpha(1 - \gamma)\frac{1-e^{-rdt}}{r}p + (1 - \alpha + \alpha\gamma)dte^{-rdt}v_1 + e^{-rdt}\hat{W}(p, T) - \hat{W}(p, T)}{dt} \bigg|_{dt \to 0} = 0
\]

(16)

After simplifying it, we get the IC constraint on the monopolist:

\[
\alpha(1 - \gamma)p + (1 - \alpha + \alpha\gamma)v_1 = r\hat{W}(p, T)
\]

(17)

To understand the intuition of the above constraint, note that the left-hand side of (17) is the immediate gain from maintaining a normal price. The gain is a composite of profits from selling to \(\alpha(1 - \gamma)\) of HU consumers at the normal price \(p\), and selling to the rest three groups of consumers at price \(v_1\) at the next instant. We construct a profit-generating process which generates a constant level of profit at every instant and yields a total payoff of \(\hat{W}(p, T)\). Equivalently, the level of the constant profit flow should be \(r\hat{W}(p, T)\), which is the right-hand side in the IC constraint. When the immediate gain equals to the equivalent constant profit flow level at every instant, which is also the immediate cost of delaying the sale, the seller has no incentive to advance or delay the sale.

We are interested in comparing the new market equilibrium with a small proportion of informed consumers and the benchmark market equilibrium without any informed consumers. The following lemma presents the first finding in our comparison:
Lemma 4  For any normal price $p$ such that $(p, T)$ satisfies the IC constraint on the monopolist with no informed consumers, and $(p, T')$ satisfies the counterpart IC constraint with informed consumers, we have $T' < T$.

We provide the formal proof in Appendix A. The intuition of Lemma 4 is: when $\gamma > 0$, the seller has a higher incentive to advance a sale as her profits from the final normal-price period, close to the sale date, are lower compared to the $\gamma = 0$ case. The reason is that all the HI consumers entering close to the sale date will not buy at the normal price. Hence, the monopolist gains less from maintaining the normal price. To guarantee that the monopolist will not deviate to advance the sale, she must shorten the normal price length so that less consumers are accumulated in the market at the end of normal price period, resulting in a lower incentive to advance the sale.

Similar to the benchmark case, to guarantee the existence of a non-trivial equilibrium, we need to prove that there exists a set of $(p, T)$ such that the IC constraint on the high-valuation consumers, the IC constraint on the monopolist, and the assumption $T \geq L$ could hold simultaneously. Note that, in the $T - p$ diagram, the IC constraint (same for both models) on high-valuation consumers (Inequation (7)) moves up when $v_h$ increases. In the meanwhile, the new IC constraint on the monopolist (Equation (17)) remains unchanged with an increase in $v_h$. Therefore, when $v_h$ is high enough, the intersection region must be non-empty. We further show that when $\gamma$ is small, the equilibrium remains at the crossing point of the monopolist’s IC constraint and the upper bound of HU consumer’s IC constraint in the $T - p$ diagram. Proposition 4 summarizes the comparison between the two market equilibria in the $\gamma > 0$ model and the $\gamma = 0$ benchmark model:

**Proposition 4** Compared to the benchmark equilibrium with no informed consumers, the new equilibrium with a small proportion of informed consumers consists of a shorter normal-price period length and a lower level of normal price, while the sale price and its length unchanged ($v_l$ and $L$ respectively).
The proof of Proposition 4 is provided in Appendix A. Figure II illustrates an numerical example that uses the same parameters as in Figure I ($v_h = 10, v_l = 4, \alpha = 0.8, r = 0.02, L = 2$) in the benchmark model.

Figure II: IC constraint for the seller with informed consumers

We compare the IC conditions of the monopolist when $\gamma = 0$ (solid red line) and $\gamma = 0.02$ (dashed red line). As we have proved before, with informed consumers, the IC constraint on the monopolist shifts up in the $T - p$ diagram. Moreover, in Figure II, the benchmark model equilibrium $(p^*, T^*)$ is represented by the crossing point of the solid red curve and the blue curve (point A). When there is a small $\gamma > 0$ fraction of informed consumers, the new equilibrium $(\hat{p}, \hat{T})$ is pinned down by the crossing point of the dashed red curve and the blue curve (point B). Compared to the benchmark equilibrium, in the new equilibrium with a positive amount of informed consumers, both normal price $p$ and the period length of normal price $T$ decrease.

The intuition is as follows. The monopolist holds a sale when there are enough con-
sumers accumulated in the market. She trades off between a higher profit from maintaining the normal price for some more time, and, a sale immediately to cash out the accumulated group of consumers as early as possible. When $\gamma > 0$, there are two forces that give the monopolist a stronger incentive to advance the sale than the benchmark case. First, she gains less by maintaining a high price. This is because HI consumers who enter the market relatively late in a price cycle will wait for the sale. Therefore, when approaching the sale, maintaining a high price can only extract the profits from HU consumers instead of all high-valuation consumers, which is the case in the benchmark. Secondly, when $\gamma > 0$, there are more HI consumers accumulated in the market, hence, the monopolist gains more by holding a sale and she is motivated to hold a sale earlier. Finally, as uninformed high-valuation consumers could expect that the monopolist has an incentive to reduce the price cycle length when there are informed consumers, the monopolist has to lower the normal price to make them purchase at the time of their arrival.

5 Welfare Analysis

In the previous sections we have solved and compared the monopolist’s optimal pricing schemes when there are no informed consumers and when some consumers become informed. There are two question left to answer: first, how does the market equilibrium change when more and more consumers become informed? Second, how those changes in consumer information, or, how the presence of price trackers affect each market player’s welfare and the total social welfare? The following proposition addresses both questions:

**Proposition 5** Within the set of $\gamma$ parameter values that satisfy both Proposition 3 and Proposition 4, when $\gamma$ increases, in equilibrium the normal price becomes smaller, and the normal-price period length becomes shorter; the monopolist profits less, while the welfare of low-valuation consumers remains unchanged and that of all high-valuation consumers increases.
We provide the complete proof in Appendix A. Similar to Lemma 4, we show that when \( \gamma \) increases, the IC constraint on the monopolist shifts up in the \( T - p \) diagram. Thus, both of the two elements in the equilibrium pricing scheme \((p, T)\) decreases with an increase in \( \gamma \). The intuition is similar to Lemma 4. When more consumers become informed, the monopolist gains less by maintaining the normal price at the time close to the sale. Moreover, a bigger \( \gamma \) implies more HI consumers waiting for the sale which further suggests an higher incentive to hold a sale earlier. However, HU consumers can expect the advance of sale, then the monopolist has to also reduce the normal price to ensure early purchases.

So far we have shown that, with a higher proportion of informed consumers, the equilibrium normal price and its period length, thus the length of the price cycle, will decrease. The question remaining is how does this change affect the welfare of different parties, including the seller, the low-valuation consumers, the HU consumers who remain to be HU after \( \gamma \) increases, the HI buyers who remain to be HI, and the new users of price tracker, who switch from HU to HI.

First, the low-valuation consumers’ welfare remains unchanged and equals to zero. No matter what the value \( \gamma \) is, the market clearing-out price is always \( v_l \). Therefore, the welfare for low-valuation consumers are always zero. Second, all high-valuation consumers are better off with a decrease in \( p \) and \( T \). A decrease in the length of the price cycle makes sales more frequent so that high-valuation consumers have a higher probability of buying at the sale price, and also the cost of waiting becomes smaller. Moreover, a decrease in \( p \) increases their welfare when they purchase during the normal-price period. For high-valuation consumers who switch from uninformed to informed, besides what we have mentioned, they enjoy an additional benefit by knowing accurately when to wait for the sale. Finally, the profits of the monopolist decrease with an increase in \( \gamma \). All three changes reduce the profits of the monopolist: a lower \( p \), a lower \( T \) and newly-informed consumers waiting for sales.

The total social welfare is ambiguous. In terms of social welfare, what matters is only the transaction timing. The earlier a transaction is, the less value loss it associates with. When \( \gamma \)
increases, there are three forces driving the social welfare towards different directions. First, a decrease in $T$ reduces the average waiting time of consumers in the market and consequently earlier purchases increase the social welfare. Second, a decrease in $p$ specifically increases $N(p)$, which implies that more HI consumers purchase earlier instead of waiting. Third, after becoming informed, the new users of price trackers could strategically choose to wait for the sale, thus delaying the transactions and harming the social welfare. The first two forces increase the total social welfare, while the third one reduces the social welfare. It is ambiguous that which force dominates.

6 Extension: Monopolist with Full Commitment

This section extends the model to solve for the monopolist’s full-commitment optimum. In the first part, we first consider the benchmark case with a change in the setting that now the monopolist has the power to make an ex ante commitment to a certain pricing scheme. In the second part, we consider the market with some informed consumers and we compare the market equilibrium with monopolist’s full commitment power to the one without commitment power.

6.1 Full Commitment with No Informed Consumers

This subsection presents the solution to the monopolist’s problem when the monopolist can fully commit to a non-stochastic pricing scheme $\{p_t\}$, where $t \in [0, \infty)$, before the entry of any consumer at $t = 0$. The marker has no informed consumer. The monopolist’s objective function and the constraint for the high-valuation consumers stay the same as in the benchmark. The main difference is that, the IC constraint on the monopolist with full commitment
does not exist anymore. Therefore, we can rewrite the optimization problem as follows:

$$\max_{\{p, T\}} \Pi(p, T) = \alpha p (1 - e^{-rT}) + (1 - \alpha) v_l r T e^{-rT} + v_l (e^{-rT} - e^{-r(T+L)}) \over r (1 - e^{-r(T+L)})$$

(18)

subject to:

$$v_h - p \geq (v_h - v_l) {1 - e^{-rT} \over rT}$$

(19)

Obviously (19) must be binding, otherwise for any \((p, T)\), the seller can increase \(p\) to achieve a higher payoff. Recall that in the non-commitment case, in equilibrium, the IC constraint on high-valuation consumers (7) is also binding. Define \(p^{ICH}(T) = v_h - (v_h - v_l) \frac{1 - e^{-rT}}{rT}\), which is an increasing function derived from (19) being binding. Following the previous section, \((p^*, T^*)\) denotes the equilibrium of the non-commitment case, and \(T^*\) is the solution of:

$$(\alpha p^{ICH}(T) - v_l)(1 - e^{-rL}) + (1 - \alpha) v_l [1 - r T - e^{-r(T+L)}] = 0$$

(20)

And \(p^* = p^{ICH}(T^*)\). In this section, we let \((p^c, T^c)\) denote the equilibrium of the full-commitment case. And \(T^c\) is determined by:

$$\frac{d\Pi(p^{ICH}(T), T)}{dT} = \left. \frac{\partial \Pi(p, T)}{\partial T} \right|_{p=p^{ICH}(T)} + \frac{\partial \Pi(p, T)}{\partial p} \frac{\partial p^{ICH}(T)}{\partial T} = 0$$

(21)

which is equivalent to:

$$(\alpha p^{ICH}(T) - v_l)(1 - e^{-rL}) + (1 - \alpha) v_l [1 - r T - e^{-r(T+L)}] + \alpha \frac{1 - e^{-rT}}{r (1 - e^{-r(T+L)})} \frac{\partial p^{ICH}(T)}{\partial T} = 0$$

(22)

To compare \(T^*\) and \(T^c\), notice that \(T^*\) is determined by the IC of the monopolist (8) crossing the IC on high-valuation consumers (7) from below on \(T - p\) diagram. Therefore, for any \(T \in [L, T^*]\), the left hand side of (20) is greater or equal to 0. Moreover, (20) and (22) differ in the third term of (22), which is strictly greater than 0. Hence, for any \(T \in [L, T^*]\), (22) cannot
hold. If (22) has a solution, it must be greater than $T^\ast$.

We also need to prove that the equilibrium $T^c$ exists. Note that when $T$ goes to infinity, the left hand side of condition (22) converges to minus infinity. As the left-hand side function in (22) is continuous in $T$, we know that $T^c$ exists, at which the function value equals to zero. Since $p^{ICH}(T)$ is increasing, we also have $p^c > p^\ast$. Proposition 6 summarizes the comparison between the two equilibrium pricing schemes without and with commitment:

**Proposition 6 (Full commitment vs non-commitment with no informed consumers)**

When the monopolist can commit to a pricing scheme and there are no informed consumers, the normal price in equilibrium becomes higher and the normal-price period lasts longer compared to the non-commitment equilibrium. Moreover, the monopolist makes a higher profit with commitment.

### 6.2 Full Commitment with Informed Consumers

A fraction $\gamma > 0$ consumers become informed of the product’s historical prices. We let $(\hat{p}^c, \hat{T}^c)$ denote the optimal pricing scheme that the monopolist commits to in this scenario. The main difference between the full commitment case and the non-commitment case is that the IC constraint on the monopolist no longer applies. Hence, the optimization problem becomes:

$$\max_{\{p, T\}} \hat{\Pi}(p, T)$$

subject to:

$$v_h - p \geq (v_h - v_l) \frac{1 - e^{-rT}}{rT}$$

where $\hat{\Pi}(p, T)$ is the objective function defined in the non-commitment case (Equation (13)).

When $\gamma$ is small, (24) must be binding, otherwise the monopolist can increase the price until (24) is binding. The reason is that, if $\gamma$ is small, the monopolist gains more from selling at a higher prices to HU consumers than her loss from more HI consumers waiting for the
sale. Therefore, \((\hat{p}^c, \hat{T}^c)\) must satisfy the following condition:

\[
(e^T - e^{-rL}) \frac{\partial \hat{\Pi}(\hat{p}^c, \hat{T}^c)}{\partial p} \frac{\partial p^{ICH}(\hat{T}^c)}{\partial T} + \alpha (1 - \gamma) p + (1 - \alpha + \alpha \gamma) v_l = r \hat{W}(\hat{p}^c, \hat{T}^c)
\] (25)

Where \(p^{ICH}(T)\) is defined as \(p^{ICH}(T) = v_h - (v_h - v_l) \frac{1 - e^{-rT}}{rT}\). When \(\gamma\) is small, \(\frac{\partial \hat{\Pi}(\hat{p}^c, \hat{T}^c)}{\partial p} > 0\) holds and \(\frac{\partial p^{ICH}(\hat{T}^c)}{\partial T}\) is always positive. Therefore, the first term of (25) is positive. Also note that this term is the only difference between (25) and (17).

To compare \(\hat{T}\) and \(\hat{T}^c\), notice that any \(T \in [L, \hat{T}]\) will induce LHS of (17) greater than the RHS of that. Since \((e^T - e^{-rL}) \frac{\partial \hat{\Pi}(\hat{p}^c, \hat{T}^c)}{\partial p} \frac{\partial p^{ICH}(\hat{T}^c)}{\partial T} > 0\) when \(\gamma\) is small, for any \(T \in [L, \hat{T}]\), (25) cannot hold. Therefore, if there exists any equilibrium with commitment, it must be lying on the boundary of (24) and at \(T > \hat{T}\) region. The existence of equilibrium is guaranteed by the continuity of \(\hat{\Pi}(p, T)\) and \(p^{ICH}(T)\). Moreover, as \(p^{ICH}(T)\) is monotonically increasing, we also have \(\hat{p}^c > \hat{p}\). We summarize the comparison in the following proposition:

**Proposition 7 (Full commitment vs non-commitment with informed consumers)**

With the monopolist’s full commitment power and a small proportion of informed consumers, the normal price in equilibrium becomes higher and the normal-price period lasts longer compared to the non-commitment equilibrium. The monopolist makes a higher profit with commitment.

7 Conclusion

In this paper we exploit a real-life setting where consumers are uninformed about the past transaction information including historical prices of a durable good sold by a monopolist. One of the questions we aim to answer is how the market equilibrium in this setting differs from the classic literature with assumption on perfect consumer information. Before answering this question, it is essential to understand what information that the historical prices contain to affect consumer purchase decisions.

Consistent with the seminal paper Conlisk et al. (1984), the monopolist in our model
implements a cyclical pricing strategy. Each price cycle ends with a sale, and the sale aims at clearing out the low-valuation consumers who choose not to pay for the price set at the time when they enter the market. After the size of the accumulated group of low-valuation consumers gets big enough, the monopolist will hold a sale as early as possible. In our benchmark model with no informed consumers, when consumers enter the market, they do not know the timing of their entry in a price cycle, thus no information on how many low-valuation consumers being accumulated. Therefore, they could not predict when the next sale is going to take place. As a result, the monopolist only sets one recurrent price besides the sale price within a price cycle. The difference compared to the classic literature is that in our model the non-sale price is a pooling equilibrium, while in the literature during non-sale period, the pricing scheme normally displays a step-function shape instead of remaining at a single value.

We then extend the benchmark model to consider the presence of price-tracker users. Those are defined as informed consumers who could observe historical prices when they enter the market. We try to answer a new question that has never been addressed in the literature: how does the presence of price trackers, like CamelCamelCamel, affects the market equilibrium?

We first find that, as informed consumers are able to figure out the accumulated number of low-valuation predecessors, they could predict the timing of the next sale. Therefore, they could optimally choose to wait for the sale if their entry time is close enough to the sale. The existence of those informed consumers results in a higher incentive for the monopolist to advance sales, and accompanied by lowering the normal price to make uninformed consumers still purchase early. The length of the price cycle becomes shorter (with the sale period length unchanged) when the proportion of high-valuation consumers increases.

After analyzing the new market equilibrium with informed consumers, we also conduct welfare analysis. The existence of price trackers, and/or the increasing trend of the number of price-tracker users, leads to lower profits of the monopolist, but benefits all the con-
sumers in the market. The total welfare is ambiguous as when the proportion of informed consumers changes, some transactions are advanced but some transactions get delayed.

Besides providing insights on the economic role of historical prices, our model could also generate important managerial implications. We predict that, as long as the number of price-tracker users are relatively small compared to the whole population of consumers, the existence of price trackers benefit consumers. However, by contrast, sellers or platforms are unwilling to disclose the historical price information because by doing so it reduces their profits, which is consistent with what is happening in reality.
References


Appendix A

Proof of (9) Increasing and Concave in $T$

Its first-order derivative with respect to $T$ is:

$$-\frac{v_h - v_l (rT + 1)e^{-rT} - 1}{r \frac{T^2}{T^2}}$$

Since $\forall x, x + 1 \leq e^x$, we have $rT + 1 < e^{rT} (rT \neq 0)$. Therefore the first-order derivative is bigger than zero, and the function is increasing in $T$.

We then take the second-order derivative, it is

$$-\frac{v_h - v_l 2 - e^{-rT} (r^2T^2 + 2rT + 2)}{r \frac{T^3}{T^3}}$$

We can first prove that function $\frac{x^2+2x+2}{e^x}$ is decreasing on $x \geq 0$. The proof is straightforward as its first-order derivative $-\frac{x^2}{e^x} < 0$. Therefore $\frac{x^2+2x+2}{e^x} < 2$ for $x \geq 0$. Hence, $-\frac{v_h - v_l 2 - e^{-rT} (r^2T^2 + 2rT + 2)}{r \frac{T^3}{T^3}} < 0$ and the function is concave in $T$.

Proof of (8) Increasing and Convex in $T$

The first-order derivative with respect to $T$ is $r - re^{-r(T+L)} > 0$, and the second-order derivative with respect to $T$ is $r^2 e^{-r(T+L)} > 0$. Therefore, function (8) is increasing and convex in $T$.

Proof of Proposition 3

We aim to show that there exists a non-empty set of $\gamma$ such that the shape of the optimal equilibrium is the same as $\gamma = 0$ case. Intuitively when $\gamma$ is small enough, reducing price for at least $L$ length to attract HI consumer is not worthwhile, since it also reduces the price during this period for the HU consumers and it generates lost.
To find a sufficient upper bound of $\gamma$, we look for the highest $\gamma$ such that any pricing scheme $\{p_t\}$ that is not the shape in the $\gamma = 0$ case is not a Nash equilibrium. That is, the seller will always have incentive to deviate all prices that is not $v_l$ to the highest price so to increase his profit. We firstly study the case where within each cycle, there are three different prices. Then we show that the exact same method can be generalized to the situations when there are many different prices. Also notice that the argument in Lemma 4 such that there must be a price $v_l$ within each period still holds here. Hence after every cycle the stock of low type consumer is totally cleared and the game is exactly as in the very beginning. Therefore, we can safely restrict our discussion within only one cycle.

Let the price within each cycle to be $\{p, p_l, v_l\}$, where $p > p_l$ is the highest price that is targeting to the HU consumers and HI consumers in the early stage of the cycle. $p_l > v_l$ lasts for at least $L$ long to aim to capture the HI consumers that comes in the middle of the cycle, and $v_l$ lasts for at least $L$ long to clear out the market. Note that we don’t require time of $p$ to be longer than $L$ here: it actually slacked the region of discussion. Hence this is sufficient for finding an upper bound. Same reasoning as before, the optimal $v_l$ will last for exactly $L$ length, and let $p_l$ lasts for $L_l \geq L$ length. Then, the total length of high price cycle is $T > L_l$, which is the total time of $p$ and $p_l$ within each cycle. Same as (12), we can define $N(p)$ as $(v_h - v_l)e^{-rN(p)} = v_h - p$. We divide this proof into two cases: $N(p) \geq L$ and $N(p) < L$.

**Case 1: $N(p) \geq L$** We start with the following lemma:

**Lemma 5** When $N(p) \geq L$, any pricing scheme in which $p_l$ starts before $T - N(p)$ is not an equilibrium.

**Proof** Assume a pricing scheme is such that $p_l$ period starts at $\hat{t} < T - N(p)$. If $L_l > L$, then we can simply deviate $\min\{T - N(p) - \hat{t}, L_l - L\}$ length of price in the beginning of $p_l$ to $p$. This change will not affect number of consumers who buy at this time interval, but only increases the price. Hence it is a profitable deviation.
If $L_l = L$, we can shift the $L$ length interval of $p_l$ to right so that it begins at $T - N(p)$. The worst case is after the shifting, all HI consumers in the interval $\hat{t} + L, T - N(p) + L$ will not buy at $p_l$ and the seller bears a pure loss during that period for HU consumers since we reduce price from $p$ to $p_l$. But the seller also gains that much from raising the price of same length of time in the beginning. All other unchanged. Since the seller discounts the future, she will get bettered off by right shifting the $p_l$ period. Hence any pricing scheme where $p_l$ starts before $T - N(p)$ cannot be an equilibrium.

**Lemma 6** If there exists a threshold $\hat{\gamma}$, such that for any $\gamma \leq \hat{\gamma}$ and $L_l = L$, any three-prices scheme cannot constitute an equilibrium. Then for any $\gamma \leq \hat{\gamma}$ and $L_l > L$, any three-prices scheme cannot constitute an equilibrium either.

**Proof** Assume $L_l > L$. Firstly, if at the right end of $L_l$ length, the HI consumer is not just indifferent between buying at $p_l$ or waiting until $v_l$, then this pricing scheme cannot be an optimal equilibrium. If the HI consumers that enter in this point strictly prefer to buy now then wait, then if this is an equilibrium, the seller can slightly increase $p_l$, which is also going to be an equilibrium and gives higher payoff. If the HI consumers that enter in this point strictly prefer to wait then buy now, the seller can deviate to a slightly shorter $p_l$ length so to avoid too much profit loss from HU consumer. Hence it cannot be an equilibrium since there is a profitable deviation. Then we discuss the case where at the end of $p_l$, HI consumer is actually indifferent between buy or wait. We can analytically write the condition where 3-price scheme is not an equilibrium:

$$\left(1 - \gamma\right)(p - p_l) \int_0^{L_l} e^{-rt} dt + \gamma v_l L_l e^{-rN} > \gamma p_l \int_0^{L_l} e^{-rt} dt$$

(26)

It is easy to see that for any $\gamma$ such that (26) holds for $L_l = L$, for the same $\gamma$ and $L_l > L$ (26) also holds. Intuitively, this is because when we are increasing $L$ by $dt$, the increment of the first term in LHS and RHS are discounted by the same weight $e^{-rL_l}$, but the increment of the.
second term is not discounted. Hence it is sufficient for us to find a threshold of $\gamma$ such that $L_l = L$ cannot be an equilibrium.

With Lemma 5 and Lemma 6, we can analytically solve for a threshold $\hat{\gamma}$ such that $\forall \gamma < \hat{\gamma}$, any three-price scheme cannot be an equilibrium. We need:

$$
(1 - \gamma)(p - p_l) \frac{1 - e^{-rL}}{r} + \gamma \frac{v_l}{v_h - v_l} \ln \left( \frac{v_h - p_l}{v_h - p} \right) > \gamma \frac{p_l}{r} \frac{p - p_l}{v_h - p_l}
$$

holds for any $p_l \in [v_h - (v_h - p)e^{-rL}, p]$ and any $p \geq v_h - (v_h - v_l)e^{-rL}$. Firstly, when $\gamma$ is decreasing and hold others unchanged, (27) is always more slack. Thus there is a cut-off point $\hat{\gamma}$ such that any $\gamma < \hat{\gamma}$ and $(p, p_l)$ satisfies the conditions above, (27) holds. Secondly, note that for a fixed $p_l$, increasing $p$ will always make (27) more slack. Thus for any $p_l$, we only need to discuss the case where $p = v_h - (v_h - p_l)e^{-rL}$. Replace it to (27), we have:

$$
\frac{1 - \gamma}{r} (v_h - p_l) (1 - e^{-rL})^2 + \gamma \frac{v_l (v_h - p_l)e^{-rL}}{v_h - v_l} > \gamma \frac{p_l e^{-rL}}{r}
$$

(28)

It is easy to see that (28) is less slack when we increase $p_l$. When $p = p_l = v_h$, any high type buyer will always optimally wait until discount. Therefore $p_l < p < v_h$ must hold for any equilibrium. And For any $p_l < v_h$, we can find a threshold $\hat{\gamma} > 0$ such as:

$$
\hat{\gamma} = \frac{(v_h - p_l)(1 - e^{-rL})^2}{p_l e^{-rL} - \frac{v_l (v_h - p_l)e^{-rL}}{v_h - v_l} + \frac{(v_h - p_l)(1 - e^{-rL})^2}{r}}
$$

(29)

and for any $\gamma < \hat{\gamma}$ the pricing equilibrium $(p, p_l, v_l)$ cannot be an equilibrium since the seller will have incentive to deviate $p_l$ part up to $p$.

**Case 2: $N(p) < L$** Similar to in Case 1, we can show that any equilibrium must be such that $L_l = L$ and $p_l$ ends exactly at time $T$. If $L_l > L$, a profitable deviation is to raise the price in the beginning of $p_l$ period until $L_l = L$. This will just make the HI consumers buy at
higher price $p > p_l$, and it is profitable. If the $p_l$ period does not end at $T$, then a profitable deviation is to move the $p_l$ period right until it ends at $T$. It is easy to show that this is also profitable.

Furthermore, note that compare to Case 1, under this case the seller has stronger incentive to deviate $p_l$ to $p$. The reason is in the $[T - L, T - N(p)]$ period, deviating $p_l$ to $p$ actually gives the seller higher payoff as even if the seller deviates to $p$ in this period of time, the HI consumers in this interval will still buy at price $p$. Also notice that in Case 1, the $\hat{\gamma}$ is actually decreasing with $p$. In Case 2, to let $N(p) < L$, $p$ has to be small. This makes the incentive of deviation condition even easier to be satisfied. Therefore, for any three-pricing scheme $(p, p_l, v_l)$ such that $N(p) < L$, there must be a positive $\hat{\gamma}$ such to make for any $\gamma < \hat{\gamma}$, the 3-pricing scheme $(p, p_l, v_l)$ cannot constitute an equilibrium.

To generalize it to any N-price scheme $\{p, p_1, p_2 \ldots p_{N-2}, v_l\}$, note that we can use the exactly same method in Case 1 to show that for any $p < v_h$, there is always a positive threshold $\hat{\gamma}$ such to guarantee that for any $\gamma < \hat{\gamma}$, seller has incentive to deviate $p_1$ to $p$. Therefore, we show that when $\gamma$ is small enough, only two-price scheme or an uniform price $v_l$ can be equilibrium.

**Proof of Lemma 4**

To compare the new IC constraint on the monopolist in section 4 with the benchmark one, we first rewrite the IC constraint in the benchmark model (Equation (8)) as follows:

$$\frac{\partial \Pi(p, T)}{\partial T} = 0 \Rightarrow \alpha p + (1 - \alpha)v_l = rW(p, T)$$

(30)

where $W(p, T)$ is similarly defined as the discounted future profits at the beginning of a sale in the benchmark:

$$W(p, T) = \frac{T(1 - \alpha)v_l r + v_l(1 - e^{-rL}) + \alpha p[e^{-rL} - e^{-r(T+L)}]}{r[1 - e^{-r(T+L)}]}$$

(31)
We subtract $\alpha\gamma(p - v_l)$ at both sides of equation (30):

$$\alpha(1 - \gamma)p + (1 - \alpha + \alpha\gamma)v_l = rW(p, T) - \alpha\gamma(p - v_l) \quad (32)$$

Let $\Delta(p, T) = rW(p, T) - \alpha\gamma(p - v_l) - r\hat{W}(p, T)$, we have:

$$\Delta(p, T) = \frac{\alpha\gamma v_l[1 - e^{-r(T+L)}] - \alpha\gamma p[1 - e^{-r(T-N+L)}] - N\alpha\gamma v_l r}{1 - e^{-r(T+L)}}$$

$$= \frac{\alpha\gamma r \int_{0}^{T-N+L}(v_l - p)e^{-rt}dt + \alpha\gamma v_l r \int_{T-N+L}^{T+L}(e^{-rt} - 1)dt}{1 - e^{-r(T+L)}} \quad (33)$$

$$< 0$$

$\Delta(p, T) < 0$ holds for any $(p, T)$. This implies that for any $(p, T)$ that satisfies (30), the new IC constraint in the model with informed consumers (Equation (17)) cannot hold, and the left-hand side of (17) is strictly smaller than the right-hand side. With $\gamma > 0$, compared to the benchmark $\gamma = 0$ case, the IC condition is shifted upward in the $T - p$ diagram. In other words, for any $p$, we need a shorter price cycle $T$ to meet the new IC constraint on the monopolist.

Note that the left-hand side of (17) is irrelevant to $T$. Therefore, we only need to show that for any $T' \geq T$, the right-hand side must be greater than the left-hand side, we are done. Note that we can rewrite the right-hand side of (17) as:

$$r\hat{W}(p, T) = \int_{0}^{L} v_le^{-rt}dt + \int_{L}^{L+T-N}(\alpha p + (1 - \alpha)v_le^{rt})e^{-rt}dt + \int_{L+T-N}^{L+T}(\alpha(1 - \gamma)p + (1 - \alpha + \alpha\gamma)v_le^{rt})e^{-rt}dt$$

$$\int_{0}^{T+L} e^{-rt}dt \quad (34)$$

For simplicity, let $A(p, T) = \int_{0}^{L} v_le^{-rt}dt + \int_{L}^{L+T-N}(\alpha p + (1 - \alpha)v_le^{rt})e^{-rt}dt$. We prove Lemma 4 by contradiction. If there exists a $T' > T$, such that $(p, T')$ satisfies (17). Then let $\delta_t =$
The first inequality is because \( \alpha p > \alpha(1 - \gamma)p \). The second inequality is because this is actually the weighted average of two parts: one part is \( WR(p, T) \) and another part is 
\[
\int_{T+T+N}^{T+N+\delta} (ap + (1 - \alpha)v_t)e^{-rt}dt + \int_{T+T-N}^{T-N+\delta} (\alpha(1 - \gamma)p + (1 - \alpha + \alpha\gamma)v_t)e^{-rt}dt
\]
\[
> \alpha(1 - \gamma)p + (1 - \alpha + \alpha\gamma)v_t = \text{RHS of (17)}
\]

Hence, we show that for any \( p \), when \( \gamma > 0 \), the IC constraint on the monopolist implies that the length of the price cycle is shorter compared to \( \gamma = 0 \).

**Proof of Proposition 4**

We aim to show that with small \( \gamma \), along the IC constraint of monopolist (17), the payoff of the monopolist is always increasing along the \((p, T)\) increasing direction. And therefore the optimal equilibrium for the seller would be in the crossing point of (17) and (14). We show a sufficient condition of this: \( \partial \hat{\Pi}/\partial T > 0 \) always hold for any \((p, T)\) defined by (17), and \( \partial \hat{\Pi}/\partial p > 0 \) when \( \gamma \) is small.

To show the first FOC condition, notice that the IC constraint of benchmark (30) is pinned down by \( \partial \Pi/\partial T = 0 \). We can rewrite this condition as following:

\[
\frac{\partial \Pi(p, T)}{\partial T} = 0 \Rightarrow ap + (1 - \alpha)v_t e^{-rT} = r\Pi(p, T)
\]

For any \( p \), let \((p, T)\) is on (30) and \((p, T')\) is on (17). By Lemma 4 we know that \( T' < T \). We want to show that for any \( T' < T \), \( \partial \hat{\Pi}(p, T')/\partial T > 0 \). We show by contradiction. If there
exists \( T' < T \) such that \( \partial \hat{\Pi}(p, T')/\partial T \leq 0 \), then:

\[
\frac{\partial \hat{\Pi}(p, T')}{\partial T} = 0 \Rightarrow \alpha p + (1 - \alpha)v_t e^{-rT'} \leq r\hat{\Pi}(p, T') \tag{37}
\]

Note that:

\[
r\hat{\Pi}(p, T') < r\Pi(p, T') \leq r\Pi(p, T) = \alpha p + (1 - \alpha)v_t e^{-rT} \tag{38}
\]

The first inequality comes from the fact that in benchmark, there is no informed consumer. So every high valuation consumer buy at the time he enters market. This yields strictly higher profit to the monopolist. The second inequality comes from Lemma 3. Therefore, we have:

\[
\alpha p + (1 - \alpha)v_t e^{-rT'} < \alpha p + (1 - \alpha)v_t e^{-rT} \tag{39}
\]

This cannot be true since \( T' < T \). Hence we prove that at any point \((p, T)\) on the IC condition for monopolist, (17), \( \partial \hat{\Pi}(p, T)/\partial T > 0 \) always hold.

Next, we show that with small \( \gamma \), \( \partial \hat{\Pi}(p, T)/\partial p > 0 \). Note that we have:

\[
r\hat{\Pi}(p, T) = r\hat{W}(p, T)e^{-rL} - \frac{[T(1 - \alpha) + Na\gamma]v_tr + v_t(1 - e^{-rL})}{1 - e^{-r(T + L)}}[e^{rL} - e^{-rL}]
= \alpha (1 - \gamma)e^{-rL} + (1 - \alpha + \alpha\gamma)v_t e^{-rL} - \frac{[T(1 - \alpha) + Na\gamma]v_tr + v_t(1 - e^{-rL})}{1 - e^{-r(T + L)}}[e^{rL} - e^{-rL}] \tag{40}
\]

The second inequality comes from plugging (17) into the equation. Then we can write the partial derivative of \( \hat{\Pi}(p, T) \) with respect to \( p \):

\[
\frac{\partial \hat{\Pi}(p, T)}{\partial T} = \alpha (1 - \gamma)e^{-rL} - \frac{\alpha\gamma v_t[e^{rL} - e^{-rL}]}{(1 - e^{-r(T + L)})(v_h - p)} \tag{41}
\]

It is easy to verify that for any \( p < v_h \), there exists an upper bound \( \hat{\gamma} \), such that for any \( \gamma < \hat{\gamma}, \partial \hat{\Pi}(p, T)/\partial p > 0 \).

To sum up, with a small \( \gamma \), both \( \partial \hat{\Pi}(p, T)/\partial T \) and \( \partial \hat{\Pi}(p, T)/\partial p \) are positive. Therefore the optimal price scheme the seller will choose is the crossing point of IC conditions for the monopolist himself and the HU consumers.
Proof of Proposition 5

Same as (15), we define:

$$\hat{W}_\gamma(p, T) = \frac{[T(1-\alpha) + Na\gamma]v_1 + v_1(1-e^{-rL}) + \alpha(1-\gamma)p(e^{-rL} - e^{-r(T+L)}) + \alpha\gamma p(e^{-rL} - e^{-r(T-N+L)})}{r[1-e^{-r(T+L)}]}$$

(42)

For any $\gamma$ and $\gamma'$ such that $\gamma' > \gamma$ and small enough to satisfy Proposition 3 and Proposition 4, the IC conditions for the monopolist under these two situations are:

$$\alpha(1 - \gamma)p + (1 - \alpha + \alpha\gamma)v_1 = r\hat{W}_\gamma(p, T)$$

(43)

and

$$\alpha(1 - \gamma')p + (1 - \alpha + \alpha\gamma')v_1 = r\hat{W}_{\gamma'}(p, T)$$

(44)

Rewrite (45) to:

$$\alpha(1 - \gamma)p + (1 - \alpha + \alpha\gamma)v_1 = r\hat{W}_{\gamma'}(p, T) + \alpha(\gamma' - \gamma)(p - v_1)$$

(45)

And let $\Delta_{\gamma, \gamma'}(p, T)$ be:

$$\Delta_{\gamma, \gamma'}(p, T) = r\hat{W}_\gamma(p, T) - r\hat{W}_{\gamma'}(p, T) - \alpha(\gamma' - \gamma)(p - v_1)$$

(46)

We have:

$$\Delta_{\gamma, \gamma'}(p, T) = \frac{\alpha(\gamma - \gamma')}{r[1-e^{-r(T+L)}]} \left[ \int_{0}^{T-N+L} (p - v_1)e^{-rt}dt + v_1 \int_{T-N+L}^{T+L} (1-e^{-rt})dt \right]$$

(47)

Therefore, $\Delta_{\gamma, \gamma'}(p, T) < 0$ for any $\gamma' > \gamma$. This implies that for any $(p, T)$ such to make (43) hold, LHS of (44) will always be smaller than RHS of (44). Follow the exact same steps of the proof in Lemma 4, for any $(p, T')$ such to make (44) hold, we have $T' < T$. Hence we show that an increasing of $\gamma$ will shift the IC constraint of the monopolist up on $T - p$ diagram.